Problem Chosen	2025 MCM/ICM Summary Sheet	Team Control Number
A		2516823

Stairs Age Like Fine Wine, Except They Don't Actually

Stairs have been used throughout human history as a fundamental construction component across the world. Bearing generations of footsteps, stairs serve as a lens into the past. By studying the patterns worn into the treads of a staircase, historians, archaeologists, and chroniclers alike can also bear witness to a staircase's insight. But, to do so requires more than just careful observation. In order to fully understand the history and pattern of a staircase, one ought to turn to mathematics.

In this paper, we introduce a model designed to crack open the secrets of staircases, such as age and information about how they were used. We first build a forwards direction model by simulating how steps wear down the staircase over time. In the absence of concrete data, we rely on probabilistic distributions to approximate use over time.

We simulate each foot with a multivariate normal distribution. We base the locations and covariance of the distribution on previous analysis done of the human walking gait (pattern). This determines the spread of the distribution. Further, we rotate the distribution in alignment with the average human walking gait. This is particularly useful as it provides a notion of direction of travel, which is important information for an archaeologist studying a staircase's purpose. This provides a realistic foundation for determining the precise location of expected wear over time.

Along with this, we constructed a function designed to predict the amount of wear. This function relied upon the material properties of the staircase along with the load that would be placed upon the tread of the stair with each step. We determined the load by sampling from a random distribution of human weights with an equally likely chance of the weight corresponding to a male or female.

We combined these two components to create a methodology for simulating staircase wear in the forward direction. We used an envelope variant of acceptance-rejection inverse sampling technique to sample values from our step distribution. From here, we applied our wear function, which outputted an expected amount of wear on our surface. By representing our surface with a mesh grid, we subtracted off the proper area in the corresponding location, resulting in a visualization of wear.

Since this process details the simulation in the forwards direction, we needed to reverse our model in order to help historians gather valuable data from worn staircases.

Utilizing photogrammetry methods (commonly available on any cell phone today, with more precise methods having dedicated instruments) to gather details on the wear depth and patterns on a given stair, our model measures the distance between the expected position of footfalls over the stair and the edge of a worn area to estimate the time since the stair was constructed. This estimation uses a Gaussian kernel density estimate to fit the empirical distribution (*i.e.*, the heightmap of worn depths) and then considers a Gaussian mixture model fit to this empirical distribution; by isolating the covariance matrix through measurements made along the principal axes, the time-dependent covariance matrix associated with each footfall can be used to invert the constituent equations for time given the dimensions of the proposed distribution.

Our model allows us to both simulate wear on stairs and interpret existing wear patterns; combined, these two factors allow archaeologists to understand how human-environmental interactions proceed over time and study the habits of ancient cultures in ways that were not previously possible.

Contents

1 Introduction

1.1 The Problem

Stairs, other ancient structures, and their stories have always been fascinating to archaeologists. Luckily, many modern characteristics can give insight into their history, as over time, the level top can turn bowed. Archaeologists are often interested in determining the following information about any particular staircase:

- How old are the stairs?
- How often were the stairs used?
- Was a certain direction of travel favored by the people who used the stairs?
- How many people used the stairs simultaneously?

1.2 Assumptions

In order to produce a model capable of analyzing all the desired factors, we made several key assumptions designed to assist with our mathematical analysis.

- Assume the material is isotropic (uniform material properties throughout the staircase and in all directions)
- There is no difference in gait across all steps of the staircase; that is, individuals will walk with the same step distribution, force, and gait. Thus, the differences between wear due to footfall traffic will be minimal across the set of treads in the staircase.
- When using a staircase, people are equally likely to lead with either foot. As such, there are roughly the same number of left and right steps on each tread. Therefore, the distribution on each step is balanced.
- Weight distributions across the population have not changed significantly over time. We used a normal distribution of recorded weights to simulate the load per step. We also assumed an equal probability of the traveler being a male or female and used the corresponding distributions.
- Archaeologists know what material was used to construct the staircase, or are able to get a solid estimate

1.3 Existing Models

To our knowledge, there is no existing mathematical framework designed to analyze and determine notable features of a given staircase. However, researchers have begun exploring individually some of the factors that can affect staircases. Some have looked into different varieties of stone (marble, limestone, travertine, granite) as flooring options and others have looked into different varieties of wood (horse-chestnut, silver fir, sweet cherry, etc) to determine how terraces and stairs will wear down over time. Although these models can help determine how the human foot can wear down a material (in theory), the data available does not give light to any factors of importance and focuses primarily on going forward in time rather than looking backwards. Although there is no literature describing exactly what we are trying to model, researchers have explored pieces of the overarching question. We will base our model on these findings and other relevant data related to material deterioration over time. We will then be able to introduce the element of simulation and reverse sampling to construct a more accurate and useful tool.

2 Background

2.1 Ancient Stairs

Although the origin of stairs is not fully known, they are present in many ancient civilizations. Some of the first staircases as we recognize them were present in Egypt around 2500 BCE [sta, 2023]. However, the concept of stairs dates back around 8000 years. Originally, they were incorporated into nature [sta, 2024]. As stairs continued to be in use in ancient civilizations, they began to develop a religious symbolism. In China around 55 BCE, Confucius wrote of stairs as a connection of earth with sky. Religious symbolism of staircases was also present with the Egyptians, Mayans, Indians, and Mesopotamians. Many believed that stairs were a way to ascend closer to divinity [sta, 2024]. These cultural connotations often informed how the staircases were built. Grandeur, material choices, and dimension were often chosen to suit the stair's purpose, and historians can greatly make use of staircases given their role within society.

Ancient staircases are understood to be primarily made of stone, wood, and mud, as those were the materials readily available and easy to use. This fact informed the basis of our model to be around stone and wood steps rather than other materials.

Moving into more modern history, staircases became very grand during the Renaissance Era [sta, 2024]. During this time, stairs lost religious significance and gained military importance. Accurately estimating the date a set of stairs was build is useful for determining important facts about its purpose.

Stairs continue to be an important architectural aspect of our society today. They continue to be a design centerpiece in both modern and older buildings. Overall, staircases have been an important architectural feature throughout history with interesting properties to explore. As such, they are very important for archaeologists in learning about the culture of their origin.

2.2 Böhme Abrasion Test and Archard's Formula

A paper detailing results from the Böhme Abrasion Test was the primary source for the data used on the wear rate of stone. It was designed to model Archard's abrasive wear formula simulating three-body abrasion wear [Z. Karaca, 2012]. Archard's well known formula is defined as follows:

$$V = \frac{kSF}{H}$$

where V is the abrasive wear volume lost (m^3), k is the wear coefficient, S is the sliding distance (m), F is the normal load (N), and H is the hardness (Pa) of the wearing material. From this model, the Böhme test "places a cubic stone specimen on a rotating steel disk, and then subjecting it to wear under the presence of standardized abrasive powder and constant normal load for a specified number of cycles” [Z. Karaca, 2012]. This is shown in Figure 1.

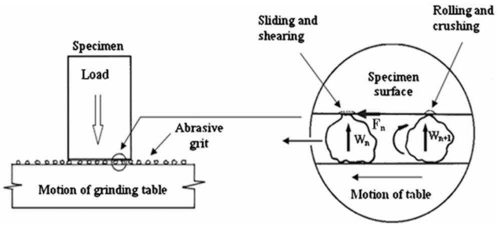

Figure 1: Wear Mechanism in the Böhme Test Apparatus [Z. Karaca, 2012]

In the literature referenced, the Böhme Test apparatus was applied to stones as a way to model foot traffic as a three-body problem. This specific test was performed using a cubic specimen (71 x 71 x 71 mm) placed on a grinding table (750 mm diameter) where abrasive grit (20 g) was spread. The specimen was then subjected to grinding for 22 revolutions under a constant load at 30 rev/min. After the 22 revolutions, the grinding table was cleaned, the specimen rotated 90 degrees and new abrasive grit placed. This was repeated for 20 cycles to reach the regression equations noted in Figure 3 [Z. Karaca, 2012]. The primary application was to examine different flooring options, which makes it easy to apply to our mathematical model.

3 Model

3.1 Measurements Needed From Archaeologist

Our model relies on the assumption that archaeologists can take non-destructive, low cost measurements. These measurements are imperative for the model as each set of stairs poses a unique problem.

The measurements required are as follows:

- the width, length, and height of each stair step on the staircase
- the volumetric quantity of material deteriorated due to wear
- the relative location of the bow (at the bottom of the staircase looking up is it on the right, left, center, or a combination) and the spread of wear (to determine upwards vs downwards motion)
- The material used to build the stair, such as type of stone

One suggested way of gathering this data is via a 3-D scan of the staircase. This is considered a low-cost, non-destructive, and simple option for obtaining data because of the large prevalence of such software, including mobile software applications.

3.2 Model Origin: Foundation Building

The first step we took was determining an appropriate methodology for modeling the distribution of step patterns. To do so, we had to develop a process that would consider the mechanics of human stair traversal.

The first parameter we studied was step width, which on average is 0.09 meters [J. Webster, 2019]. We used this to find the most realistic placement of the multivariate normal distributions. It is imperative that this parameter remains consistent across all staircases because wider or larger staircases do not correlate with a wider or larger step width. We used one multivariate normal variable per foot and took the normalized sum to determine the probabilistic location of a step. Noting that the average foot length is 0.268 meters, we scaled covariance accordingly [Daniel M. T. Fessler, 2005] such that our distribution would reflect a realistic region upon which we can apply our wear function.

In order to distinguish between upwards and downwards traversal of the stairs, we rotated each of our separate multivariate normal distributions by the average outturn of a step. We found that the average outturn is roughly 8 degrees [Morton, 1932]. When going up or down a set of stairs, we identify differences in the spread of the wear that originate from the outturn in the standard human gait. We note that combinations of traffic patterns produce different patterns of wear, and this was the primary factor we invoked to separate directions of travel.

Finally, we balanced the covariance matrix to account for the fact that there exists the higher variance in foot size compared to step width. This ultimately meant a larger variance in the y-direction.

To simulate wear in different parts of the stairs, we constructed distributions for step patterns corresponding to use on the left, center, and right portions of the staircase. Along with this, we took measures to ensure that the distribution would still work well with staircases and steps of varying sizes.

The resulting distribution is shown in Figure 2. These results were used in our function to determine the location of where to apply our wear equation, allowing for variation in the stair dimension, direction of traffic, and length of traffic.

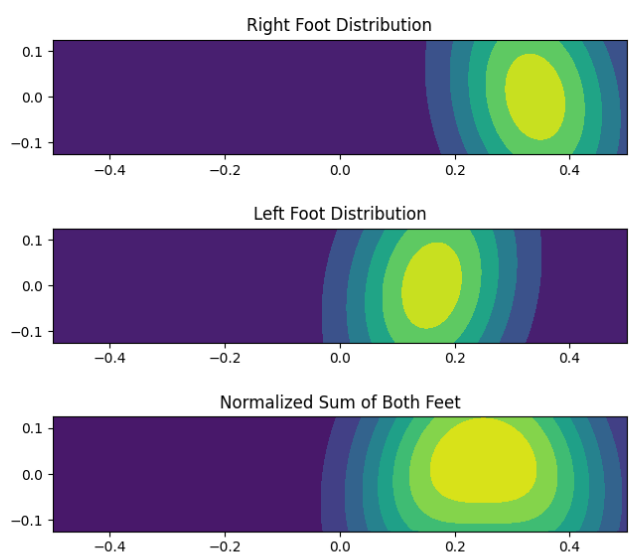

Figure 2: Probability distribution of predicted step location going down the right side of a 1 m by 0.25 m example step.

Along with our distribution, our simulation of a step was designed with a three dimensional mesh grid. This grid was given a specific resolution and dimension, so we had the ability to sample a point, apply our wear function, discussed in Section 3.5, and calculate the result in such a way that we obtained a numerical value to remove from our mesh. When the sampling process was repeated up to millions of times, we stored the results on the mesh grid and used it to understand visualization and volumetric loss.

3.3 Step Design and Simulation

The steps are represented as a three-dimensional box $S \subset \mathbb{R}^3$ with a walkable surface $W \subset \partial S$ on the top face of the box. To programatically perform operations on the box (as discussed in Section 2.2), S is discretized using a uniform mesh with grid spacings Δx, Δy, Δz. We note that $\Delta x = \Delta y = \Delta z$ is not necessarily required. W is centered at $(0, 0)$ to make representation of left-hand and right-hand worn paths symmetrical.

3.4 Step Distribution

The step distribution on a given stair is given by a Gaussian mixture model defined over W. These models are derived from a number of parameters:

- Stair traversal direction (ascent, descent, both)
- Wear area (left, right, center)
- Duration of stair usage
- Foot rotation, or outturn
- Foot dimension, including average width and length

Each individual bivariate distribution corresponds to a single foot placement on the stair. The random variables for each foot follow a Gaussian distribution

$$X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^2$ is the mean of the distribution and $\boldsymbol{\Sigma} \in \mathbb{M}_{2,2}(\mathbb{R})$ is the covariance matrix. Note that $\boldsymbol{\Sigma}$ is positive definite, meaning it can be decomposed as

$$\boldsymbol{\Sigma} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{\mathrm{T}} \tag{2}$$

where $\boldsymbol{Q} \in \mathbb{M}_{2,2}(\mathbb{R})$ is orthonormal and $\boldsymbol{\Lambda} \in \mathbb{M}_{2,2}(\mathbb{R})$ is diagonal. The diagonal entries of $\boldsymbol{\Lambda}$ (the eigenvalues of $\boldsymbol{\Sigma}$) are the standard deviations of X along its principal axes; therefore, one of these eigenvalues is associated with the length of the foot, while the other is associated with the width. In this model, these entries are time dependent, with the standard deviations increasing over time; this is to account for the assumption that a large number of people traversing the same set of steps over a short period of time will generally follow the same path, while a small number of people traversing the same set of steps over a long period of time may very somewhat more in where they step. $\boldsymbol{\Lambda}$ is therefore given by

$$\boldsymbol{\Lambda} = \begin{bmatrix} f(t) & 0 \\ 0 & g(t) \end{bmatrix} \tag{3}$$

with $t \in \mathbb{R}^+$. $\boldsymbol{Q}$ is determined by the rotation angle of the feet in the standard human gait and the direction of traversal (ascending vs. descending). The average outturn angle of human feet is roughly 8°; therefore, this nonzero angle can help visually distinguish the direction of traversal [Morton, 1932]. $\boldsymbol{Q}$ has the general form

$$\boldsymbol{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{4}$$

where the parameter θ depends on the foot (right or left) and the direction of traversal, as each of these cases will have a different traversal angle.

f and g are functions of the form

$$f(t) = w + k_f \tanh(m_f t) \tag{5}$$

$$g(t) = l + k_g \tanh(m_g t) \tag{6}$$

where w is the average width of a human foot, l is the average length of a human foot, and k_f, m_f, k_g, and m_g are scaling factors corresponding to wear rates over time. The selected values for w and l are dependent on the region of the world the staircase is found in (since these parameters vary significantly by ethnicity across the globe) [Ales Jurca, 2019]. m_f and m_g are dimensionless, while k_f and k_g have units of distance (for consistence with other units in this paper, meters, although if an archaeologist were to utilize this model with a different set of units, this would change).

The values for $\boldsymbol{\mu}$ were selected based on the portion of the stair most often climbed (left, right, center) and the average distance between feet. Given two feet distributions centered at $\boldsymbol{\mu}_L$ and $\boldsymbol{\mu}_R$, the distance between the feet, the step width $\|\boldsymbol{\mu}_L - \boldsymbol{\mu}_R\|$, was found to be 0.09 m [J. Webster, 2019]; the values of $\boldsymbol{\mu}_L$ and $\boldsymbol{\mu}_R$ were selected somewhat arbitrarily as a fraction of the step width x. Note that, in practice, this value of x would be measured for a given staircase and provided to the reverse model (see Section 3.7).

For a given random variable X, the probability density function (PDF) of X, $p_X(\boldsymbol{x})$, is given by

$$p_X(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi|\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \tag{7}$$

For a sequence of individual foot random variables $X_1, X_2, \ldots, X_n$, the PDF of the overall distribution from which footsteps on the step are sampled is

$$p(\boldsymbol{x}) = \sum_{i=0}^{n} p_{X_i}(\boldsymbol{x}) \tag{8}$$

Note that this PDF is *not* equivalent to the PDF of

$$X \sim \mathcal{N}\left(\sum_{i=0}^{n} \boldsymbol{\mu}_i, \sum_{i=0}^{n} \boldsymbol{\Sigma}_i\right) \tag{9}$$

and the random variable representing the overall placement of feet on the step, F, is not Gaussian.

3.5 Material Wear

After considering a variety of factors affecting the distribution of steps, we needed to construct a function that would predict the amount of wear on the steps for a given amount of activity. To do so, we needed to consider the hardness and durability of the material of the stairs, the weight and load of the person traveling the stairs, and the amount of steps taken.

To determine the wear rates on different materials, we turned to the literature. We were able to find data on how stone is worn down when a force similar to a foot walking is applied to it as seen in Figure 3. These regression equations demonstrate the linear relationship between the load and volumetric loss. One example model for L2, or Karia Cream Limestone is shown in Figure 3 from [Z. Karaca, 2012].

Data is measured in terms of AWR, or abrasion wear rate. AWR is calculated as

$$AWR = \frac{V_i - V_f}{A}, \tag{10}$$

where V_i, V_f are the initial and final volumes respectively and A is the surface area. As such, AWR has units of volume (cm^3) over surface area ($50cm^2$). The data collected includes AWR values for several types of Marble (M), Limestone (L), and Travertine (T). Note that in this paper, granite was also modeled but as the relationship between load and AWR was not linear, it was more difficult for the authors to reach a conclusion.

Regression models showing relationship between applied load and AWR.

Stone code	Regression equation	Coefficient of determination (R^2)
Ml	AWR = 0.0776L + 2.449	0.9924
M2	AWR = 0.0805L + 6.911	0.9904
M3	AWR = 0.0738L + 5.237	0.9741
L1	AWR = 0.0345L + 4.737	0.9789
L2	AWR = 0.0423L + 2.784	0.9699
L3	AWR = 0.0501L + 1.572	0.9668
T1	AWR = 0.0980L + 8.585	0.9813
T2	AWR = 0.0583L + 5.946	0.9965
T3	AWR = 0.0697L + 9.453	0.9884

*AWR is abrasive wear rate ($cm^3/50\ cm^2$) and L is the applied load (N).

Figure 3: From [Z. Karaca, 2012], this figure shows the regression equations for abrasion wear rates (AWR) as a linear function of load applied in Newtons

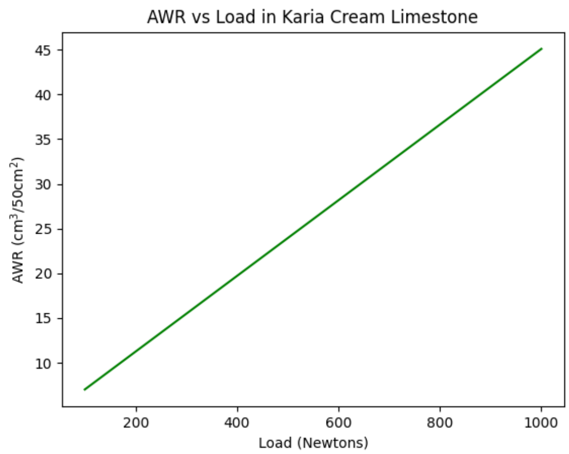

Figure 4: Wear Curve of Sample Limestone with Example Loads from 100 to 1000 Newtons. Note the linearity of it from the regression equation.

In addition to exploring how stone is affected, we also looked at how different woods are affected by abrasion. Wood is a difficult material to quantify because it is not inherently isotropic and therefore, given our assumptions, we assume a standardized wood grain in a consistent direction in order to apply and evaluate the data. We can use the lowest grain size to most closely simulate a human foot. Although wood was not explored in great detail in this paper, further research can be done to expand our model to wood, mud, and other materials.

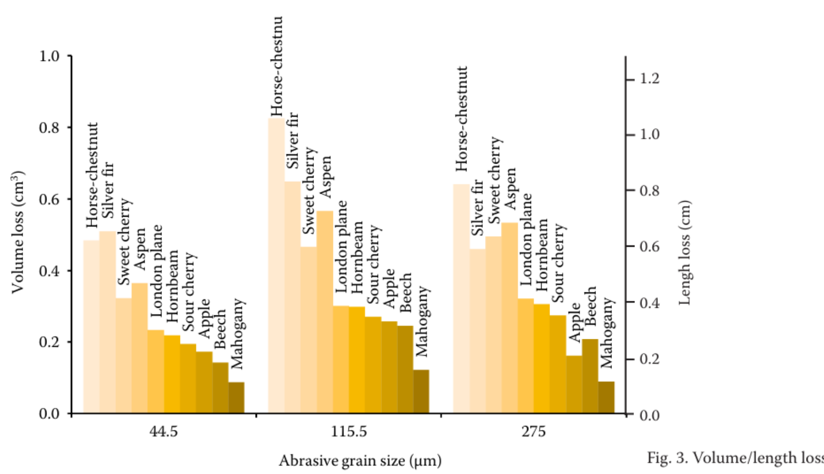

Figure 5: From [Brožek, 2017], this figure shows the volume lost in cm^3 when different abrasive grain sizes are applied. For our model, we use the lowest grain size to most closely simulate a foot.

The result of this analysis is a function $F(l, m)$, where l is the load in Newtons and m is the material properties dictating the regression curve used for determining the rate of wear.

We determined l by taking two normal distributions for the average weights of men and women. We simulated an equal chance of either a male or female walking on the staircase and sampled from the proper distribution to obtain an appropriate load.

Our model makes use of this by taking a given sample from our probability distribution and calculating a theoretical amount of material lost based on the regression equations. To do so, we take input load, which we simulated with a normal distribution of human weight. After this, using the predicted volumetric loss from the sample, we removed the corresponding amount from our stair.

We then repeated this process and re-ran our function for a set number of samples to simulate how many steps were taken overall. Since these experiments were conducted with 22 set rotations, we estimated 22 steps per sample.

3.6 Time-Forward Evolution Algorithm

Given an initial step S of dimensions x, y, z with a footstep distribution represented with a random variable F, the algorithm for producing a wear pattern on the step over a given time period $[0, t]$ is as follows:

1. *Sample from the PDF of F.* This sampling is done using an adaptive rejection sampling method in which the prior distribution Q is Gaussian with a mean equal to the average of the means of the individual foot random variables X_i and covariance matrix equal to the sum of the covariances of the individual foot random variables. The PDF $p_Q(\boldsymbol{x})$ of this random variable is then scaled to yield a new PDF $p_P(\boldsymbol{x})$ associated with random variable P according to

$$p_P(\boldsymbol{x}) = \frac{\max_{\boldsymbol{x} \in W} p_F(\boldsymbol{x})}{\max_{\boldsymbol{x} \in W} p_Q(\boldsymbol{x})} \tag{11}$$

which is guaranteed to satisfy

$$p_P(\boldsymbol{x}) \geq p_F(\boldsymbol{x}) \tag{12}$$

for all $\boldsymbol{x} \in W$. A second random variable Z is created distributed uniformly according to

$$Z \sim U(0, k) \tag{13}$$

where k is an upper bound on $p_F(\boldsymbol{x})$ over W. A large number of samples $\boldsymbol{s}$ are taken distributed according to P; an equal number of samples u are taken distributed according to Z. Samples which satisfy

$$u < p_F(\boldsymbol{s}) \tag{14}$$

are kept, which those which do not are discarded. Since F does not have a closed-form cumulative distribution function (CDF), adaptive rejection sampling is the method of choice for sampling from its distribution.

The samples gathered from F represent footsteps on the stair over time.

2. *Sample from the PDFs of X and Y.* For each footstep on W, an associated weight w is required to determine the wear on the stair. This weight is sampled from the PDFs of X and Y, the random variables representing the weights of women and men respectively. These variables are treated as standard normal variables using given data for the means and standard deviations of observed weights in the population. Whether to sample from X or Y is random with equal probability for both.

3. *Calculate the volumetric material loss.* Using X or Y, we calculate the load and use the regression equation specified by the material, as documented in Figure 3. This relies upon the material hardness. We scale our results from standard AWR units to our representative sample of the step.

4. *Remove material from the step.* After determining the amount of material to remove, return to the mesh representation of S and remove the correct amount of material. Numerically, this is done by changing the z values of W to reflect deviation from the smooth upper surface of S; the amount of layers removed from the mesh is dependent on the z-resolution of the mesh, the surface area of the foot, and the amount of material removed according to

$$n = \frac{V}{A\Delta z} \tag{15}$$

where n is the number of layers removed, V is the volume of material removed, A is the surface area removed, and Δz is the z-resolution of the mesh.

3.7 Time-Reverse Algorithm

Given a three-dimensional scan of a step provided by an archaeologist, along with the dimensions of the stair to provide scale, the algorithm for determining the age of a stair is as follows:

1. *Determine kernel density estimate (KDE) of empirical distribution.* Using the three dimensional scan of the stair, generate a map of z-heights corresponding to the empirical PDF of the step wear; from this map, use a Gaussian kernel to generate a KDE of the bivariate distribution and find an isoproportional contour at a chosen high percentile (ideally around 0.9889, the 3σ-cutoff for a bivariate distribution as calculated from the cumulative density function (CDF) of χ_2^2, the chi-squared distribution with two degrees of freedom).

2. *Determine the distance between the mean foot position of one foot in the distribution and the isoproportional KDE contour.* Assuming that the distributions of footfalls are evenly centered about the center of the empirical distribution with a 0.09 m separation, calculate the distance between the center of one foot to the KDE contour along the principal axis of the foot associated with the width and opposite the direction of the other foot. The choice of direction (away from the other foot) is to minimize the impact of the distribution of the second foot on the distribution of the first foot; this allows for the subsequent inversion step to be based mostly on one distribution and therefore yield more accurate results.

3. *Find the 1σ-value for the empirical distribution assuming a Gaussian distribution.* Given the measured distance and chosen percentile from the previous two steps, divide the distance by the σ level to determine the 1σ distance in the principal direction of the foot associated with

the width (*e.g.*, if 0.9889 was chosen [corresponding to 3σ for a bivariate normal distribution], divide the distance by three).

4. *Invert $f(t)$ for t given the 1σ-distance.* Since the covariance matrix $\mathbf{\Sigma} = \boldsymbol{Q}\mathbf{\Lambda}\boldsymbol{Q}^{\mathrm{T}}$ is dependent (via $\mathbf{\Lambda}$) on time, given the 1σ-value (corresponding to the eigenvalue of $\mathbf{\Sigma}$ given by $f(t)$), the formula for this value can be inverted to yield t.

3.8 Simulation Model Results

To assist with the visualization of our model, we compiled a set of results based off of a standardized set of parameters. This helps portray the differences in the various ways that the input parameters might affect a staircase.

Firstly, we considered a base model. This model has parameters as follows:

Variable	**Value**
Step width w	0.01 m
Step length l	0.028 m
Width scaling factor k_f	0.5
Width scaling argument m_f	0.25
Length scaling factor k_g	1
Length scaling argument m_g	1
Material	Karia Cream Limestone (L2)
Number of Samples	5,000,000
Time t	1
Direction	Up
Position	Right Side

The resulting wear is shown in 6. As we change the parameters, we show the changes in the resulting distribution in Figures 7 to 11.

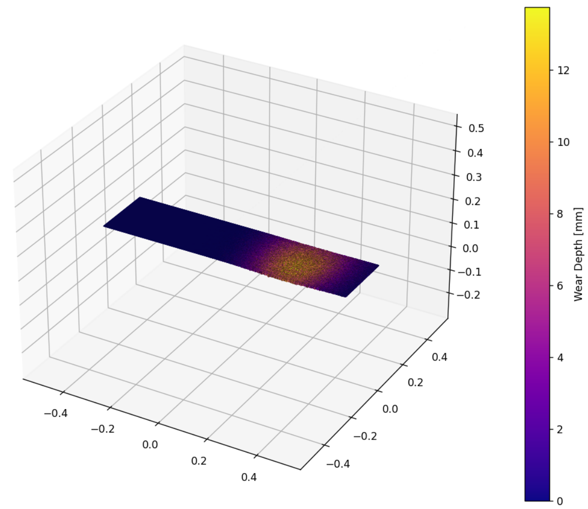

Figure 6: Sample Wear Distribution from Simulation Approach

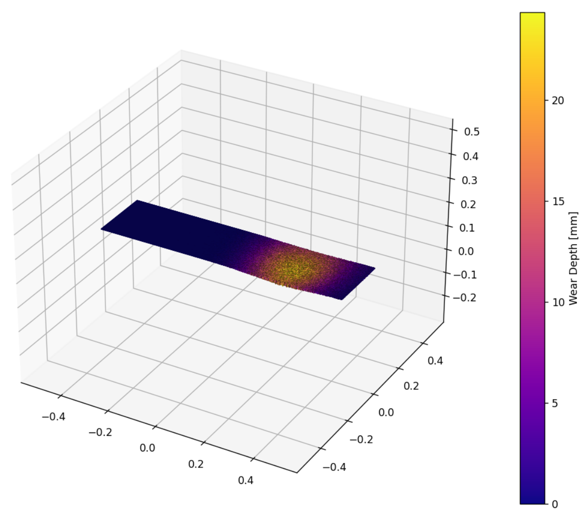

Figure 7: Changing the sample number from 5,000,000 to 10,000,000. We observe more wear here.

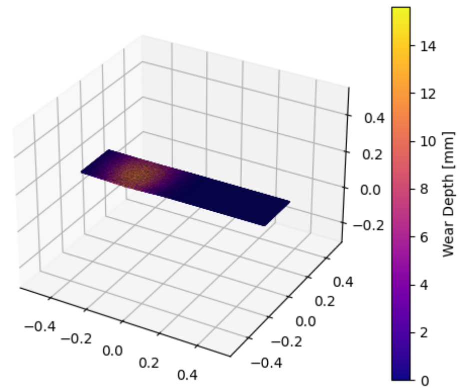

Figure 8: Changing the Position of the primary traversal to the other side of the staircase. This is an obvious change.

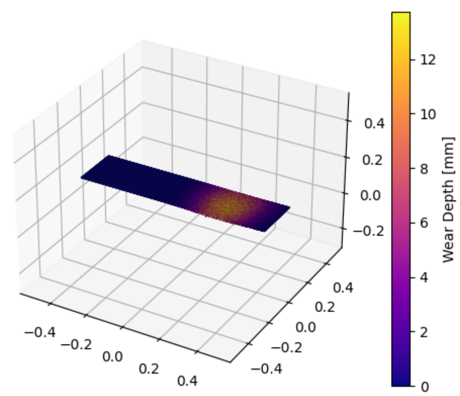

Figure 9: Changing the duration of time over which the same number of samples occur to double its previous value. We note that the spread of the distribution is slightly wider here.

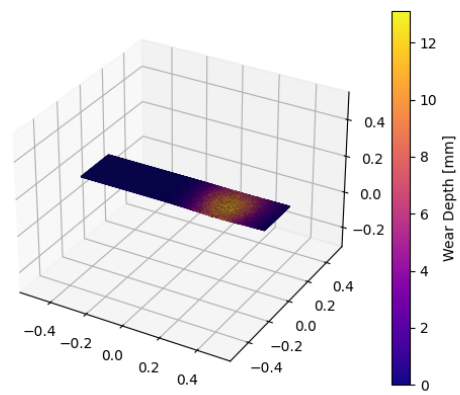

Figure 10: Changing the material to the much softer Travertine. Since it is less durable, we see more wear with the same amount of time and samples.

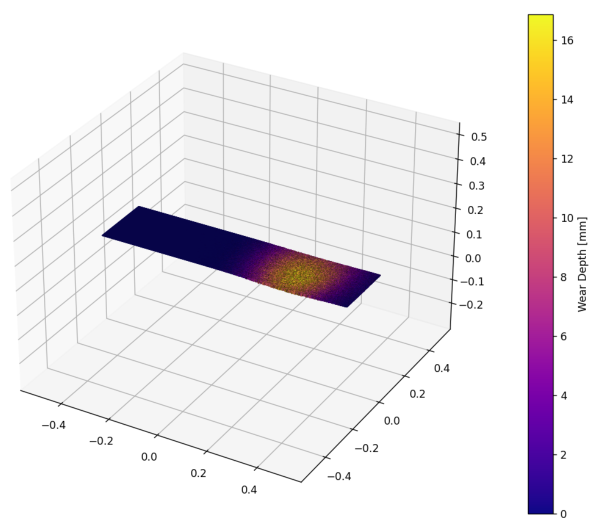

Figure 11: Changing the direction of travel from only up the stairs to both ways. We notice a change in the spread resulting in more even wear, and a bit more wear in general.

3.9 Answering Key Questions: Model Interpretation

The primary questions our model was centered around included frequency of use of the stairs, direction of travel, and simultaneous use. We address how our model provides insight to each of the matters. Further, our model can suggest other useful properties of the stairs, and so we elaborate on how these can be determined and used by archaeologists.

3.9.1 How often were the stairs used?

Frequency of use is determined from the wear on the steps. We consider two main factors when approaching this question. Firstly, we consider how worn the steps are in general. More wear (on the same material) corresponds to more use. To determine the specific frequency of use, we analyze the spread of the distribution. Over time, we expect distributions to widen horizontally. We discuss this further in Section 3.10.1. Generally, by combining observations relating to the total wear and spread of wear, we can obtain reasonably accurate estimations of the relative use. It is important to note that these estimations cannot provide an exact number or perfect daily load, but rather serve as a relative comparison.

3.9.2 Was a certain direction of travel favored by the people using the stairs?

Direction of travel is distinguished in our model by the angle of the outturn with each step. To see the differences most visibly, we compare the step distributions.

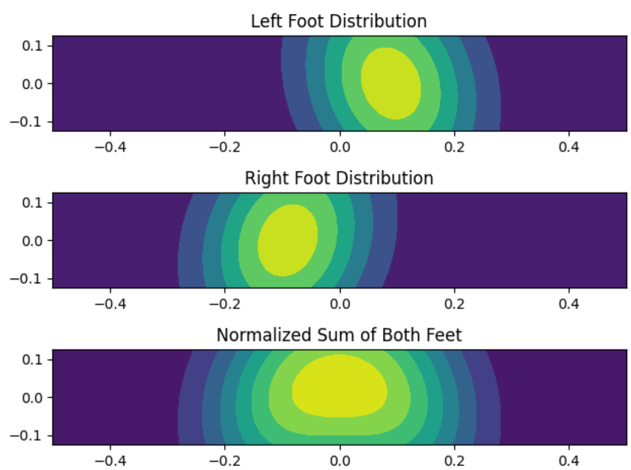

Figure 12: Wear distribution traversing down the center of a staircase.

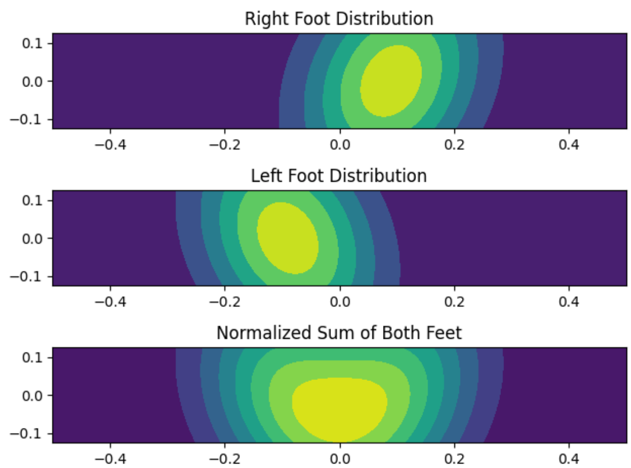

Figure 13: Wear distribution traversing up the center of a staircase.

When we invert the model, we use the wear on the stairs to create an empirical distribution, and the specific spread of this distribution will match either an upwards or downwards direction. Notably, if a distribution appears to be rather uniform, this is a clear indication of traversal in both directions. If the distribution is skewed in either direction, we expect a skew in that very same direction of travel.

3.9.3 How many people used the stairs simultaneously?

Simultaneous use of the stairs is another observable result from the spread of wear. For example, if we see one main wear pattern, such as in Figure 13, we would expect a primarily single file traversal of the staircase. If, instead, we saw two distinct wear patterns on either side in the same direction, we would expect it to be much more likely that it was traversed by two people side-by-side. An example distribution is given in Figure 14.

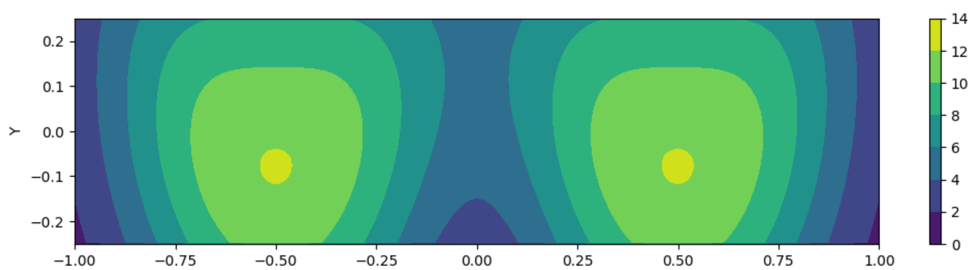

Figure 14: Example distribution of a stair used primarily for side-by-side upwards travel.

3.10 Secondary Questions

Beyond these observations, we also study several other important factors that could provide further insight and assist with archaeology. We consider a few of the questions below.

3.10.1 What information can be determined with respect to the numbers of people using the stairs in a typical day? Were there large numbers of people using the stairs over a short time or a small number of people over a longer time?

To answer this question, we study the spread of the distribution. Our model uses appropriate scaling factors in our functions for $f(t)$ and $g(t)$ from Section 3.6 to determine a relative relationship between the time the staircase was in use and the spread of the distribution, which we took to be separate from the total amount of use the staircase underwent. Thus, the total spread would be the key factor in determining the derivative of use.

3.10.2 What is the age of the stairwell and how reliable is the estimate?

To find an accurate answer for this question, there are two factors that we would need to consider. We would need to asses the balance between total wear, corresponding to samples in a simulation, and total spread of wear, corresponding to use over time. As explain in 3.10.1, we associate a wider distribution with a longer period of time of use. Thus, if we desired accurate results, we would need a case study of a staircase with a reliable age estimate that we could compare the model to. We would need to determine a precise relationship rather than a relative one, which requires comparison to a different, known example. Assuming this criteria is fulfilled, we believe that there could be steps taken to expand this model to sufficiently answer these types of questions.

3.10.3 Can the source of the material be determined?

The source of the material would depend on material properties. If the stair material is determined to have a certain set of durability properties as predicted in the model and these properties could be measured from a suspected source, a comparison could be made. However, an archaeologist must apply careful skepticism and rely upon several sources of theory and confirmation before making such an assertion. We do not recommend our model as a perfect predictor of this potential.

4 Conclusion

4.1 Sensitivity Analysis

We tested the sensitivity of our model by varying the input and observing the output. We performed a standard sensitivity test to our forward direction model, as this determines the structural integrity. We tested several staircases with slightly varying parameters and compared the output.

First, we compared the effect of sample size. We expect and desire that changes in the sampling rate will have a roughly linear relationship with the amount of wear obtained. We tested for this by running our model on a 1×0.25 meter stair. We simulated the effect of wear only going up a staircase on the right side. Here, the samples, our independent variable, were varied from 1 million samples to 2 million. As expected, and as shown in Figures 15 and 16, the change in wear was approximately linear. We observe approximately double the wear with double the samples, and we note the volumetric loss is about double as well.

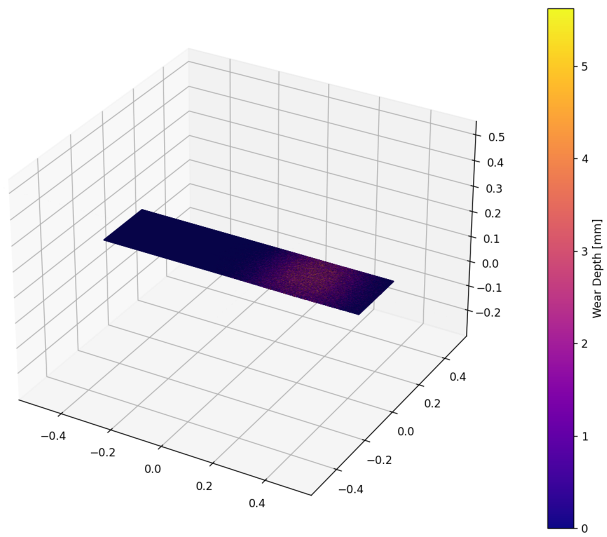

Figure 15: Sensitivity Analysis Test 1: 1 million samples

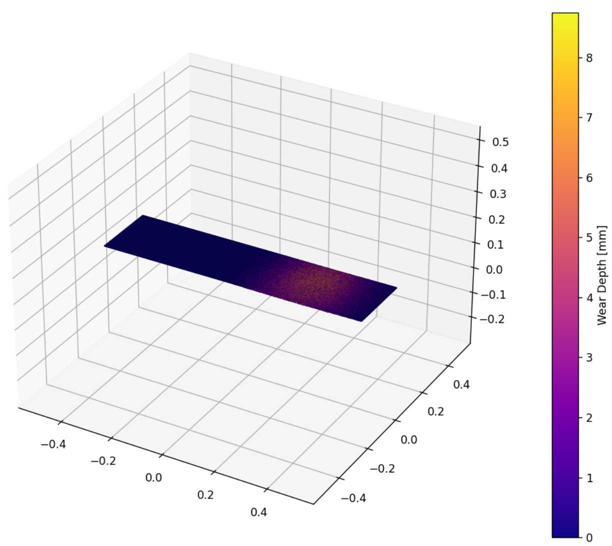

Figure 16: Sensitivity Analysis Test 1: 2 million samples

We repeated this test with several other inputs, including time, scaling factors, and foot dimension. We observed no issues with sensitivity leading to unexpected behavior; we continually observed the same amounts of volumetric loss. This improves our confidence in our model's ability to consistently and fairly be applied to unique staircases.

4.2 Model Strengths

Our model's primary strength is the customization that we offer. The model takes in a variety of parameters to output likely information about the staircase. Since there are many regional and historical differences, customization of the model is necessary to achieve our desired accuracy. For example, since we take in foot length as a parameter, we could adjust it if necessary. In Asia, foot sizes are statistically significantly smaller than those in North America or Europe. As such, we could adjust our input distribution to produce a more realistic model.

Consider another example. For our base model, we used the assumption that an equal number of men and women used the staircase. However, if archaeologists know from other societal factors that women were more likely to use a particular set of stairs, then the weight and foot size factors can be modified easily to account for this historical information.

Additionally, we can also modify our model based on the material used. Limestone, as mentioned above, was used as our default testing material, but we can use any other material with known properties as well.

4.3 Model Weaknesses

Our model does not take into account the location of the stairs, including considerations related to the climate of the location. As such, for staircases exposed to the elements, our wear does not take into consideration processes like erosion from weather phenomena. Further, we do not study the possibility of weather events impacting the wear, such as water damage. We also do not consider how climate impacts the durability of the material. We do not address how a humid environment could change material properties or how a set of stone stairs exposed to cold experiences ice wedging. These considerations are important, and given more time, would be a useful and important addition to our model since they align nicely with a simulation approach.

Our model only considers rectangular stairs. Many ancient stairs, especially from the Renaissance period, are spiral staircases, which are not compatible. Walking gait significantly changes on these types of staircases, which would require a separate model and a new step distribution.

4.4 Closing Remarks

Being able to analyze societal properties from worn down stairs could give archaeologists much needed insight into the past. Through our probabilistic model, archaeologists can begin to answer key questions about how often the stairs were used, what direction of travel was favored, how many people used the stairs simultaneously, etc. Staircases lead to a doorway into the past and our model allows archaeologists to reach the doorway to discovery.

Our model, thanks to its customizability, allows for a knowledgeable archaeologist to draw reasonable conclusions without requiring invasive or destructive methods; furthermore, it could potentially be used to predict wear on stairs in new buildings, thereby helping civil engineers and architects design safer and more resilient structures.

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